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Fractional Integral Formulation of Constitutive Equations of Viscoelasticity

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A fractional integral viscoelasticity model is discussed. The model has the same constitutive advantages as the fractional derivative viscoelasticity model. Both models need few parameters to model the weak frequency dependence of many engineering materials. Also, both models represent a causal relation between excitation and response. The fractional integral model, however, gives a unique relation between excitation and response, whereas the fractional derivative model needs initial conditions to give a unique relation. The fractional integral model is incorporated into structural equations of motion. A time-discretization scheme for the solution of these equations is outlined, and some examples are given.

Nomenclature

b = relaxation constant
C_0 = applied force amplitude
D = fractional differentiation and integration operator
$\bar{\mathrm{D}}$ = distributional fractional differentiation operator
E = instantaneous viscoelastic modulus
$E_{\mathrm{rel}}(t)$ = stress relaxation modulus
E_α = α-order Mittag–Leffler function
E_∞ = relaxed viscoelastic modulus
F_0 = applied force amplitude
$\mathcal{F}[\cdot](\omega)$ = Fourier transform
$f_i(t)$ = memory kernel
$f_\alpha(t)$ = memory kernel corresponding to fractional calculus model
k = spring constant
k_∞ = long-time spring constant
$\mathcal{L}[\cdot](s)$ = Laplace transform
M = mass
p_k, q_k = viscoelastic material parameters
$R(t)$ = applied force
$r(t)$ = spring force
s = Laplace domain variable
t = time
$u(t)$ = displacement
v_0 = velocity amplitude
α, β = fractional integral and derivative exponents
Γ = gamma function
ΔE_i = viscoelastic relaxation strength
Δt = time step
$\delta(t)$ = Dirac delta distribution
$\varepsilon(t)$ = strain
ε_0 = strain constant
$\sigma(t)$ = stress
σ_0 = stress constant
ω = Fourier domain variable and angular frequency

I. Introduction

INELASTICITY or energy dissipation effects in structural dynamics are often modeled in the frequency domain. Many of these models have weak physical significance. The most used models are the so-called hysteretic and viscous models. Both of these models are in some sense practical to use, but are physically inappropriate in one way or another, e.g., the hysteretic model is noncausal. Although time-domain viscoelastic models are available, they are not widely used, partly because these models classically demand a high number of parameters to accurately describe the observed material behavior.

Many engineering materials show a weak frequency dependence of their dynamic properties over a broad frequency range; see, e.g., Ref. 1. This weak frequency dependence is difficult to describe with classical viscoelasticity. By introducing fractional-order derivative operators instead of integer-order derivative operators in the constitutive relations of linear viscoelasticity, the number of parameters required to accurately describe the dynamic properties can be significantly reduced. The simplest uniaxial model contains only one single fractional derivative operator acting on both stress and strain. This model is commonly referred to as the fractional derivative model of viscoelasticity. It has successfully been fitted to experimental data over a broad frequency range for several polymers using only four parameters.[2] Because this model contains (fractional) derivative operators, it requires initial conditions to give a unique relation between stress and strain. Further, when this model is incorporated directly into the structural equations of motion a time differential equation of noninteger order higher than two is obtained. One consequence of this is that initial conditions of fractional order higher than one are required for solving the resulting equations of motion.[3] To avoid the problems connected to the increased order of the structural equations, Enelund and Olsson[3] used an equivalence to the fractional calculus model of viscoelasticity containing a convolution integral. The resulting equation of motion is then an integro-differential equation of order two in time. A time-discretization scheme for solving the resulting equation is presented in Ref. 4 and later generalized to three-dimensional states by Enelund and Josefson.[5]

Another formulation of the constitutive equation that is believed to be potentially useful for the application to structural dynamics is achieved by introducing fractional integral operators rather than fractional derivative operators. The resulting constitutive equation will then represent a unique relation between stress and strain. Because no derivative operators are introduced, there is no need to increase the order of the structural equation when incorporating the constitutive behavior. A constitutive relation of linear viscoelasticity involving fractional integral operators is presented by Glöckle and Nonnenmacher.[6] This model was later discussed by Friedrich.[7]

In the present study a form of the fractional calculus model of viscoelasticity containing a single fractional integral operator acting on both stress and strain is presented. The fractional integral form is compared to the forms containing a fractional derivative operator or a convolution integral regarding uniqueness and admissible domains.

Further, the incorporation of the fractional integral model into the framework of structural dynamics is presented and analyzed. Finally, a time-discretization scheme for solving the resulting

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equations of motion involving the present fractional integral constitutive equation is outlined, and some numerical examples are given.

II. Constitutive Equations of Linear Viscoelasticity

Different constitutive equations are employed to describe the behavior of viscoelastic materials. The uniaxial stress–strain relationship is often written in the time domain by use of integer-order derivatives:

$$\sum_{k=0}^{m} p_k \frac{\mathrm{d}^k \sigma}{\mathrm{d}t^k}(t) = \sum_{k=0}^{n} q_k \frac{\mathrm{d}^k \varepsilon}{\mathrm{d}t^k}(t) \tag{1}$$

Here, as in the whole study, isothermal conditions are assumed and only uniaxial stress states are considered. Equation (1) is a differential equation, and it naturally needs initial conditions. Fung[8] has shown that these conditions are

$$\sum_{i=0}^{j} p_i \frac{\mathrm{d}^{i-1} \sigma}{\mathrm{d}t^{i-1}}(0^+) = \sum_{i=0}^{j} q_i \frac{\mathrm{d}^{i-1} \varepsilon}{\mathrm{d}t^{i-1}}(0^+)$$
$$j = 1, 2, \ldots, \text{maximum} \tag{2}$$

Here maximum is the largest of m and n. Material parameters p_k or q_k in Eq. (2) not present in Eq. (1) are set to zero. Physically the conditions in Eq. (2) mean that there is a direct relation between stress and strain, i.e., there cannot initially exist a strain or a strain derivative without there being any stress or stress derivatives. There is reason to let the highest order of differentiation in Eq. (1) be equal, $m = n$, assuming also that the highest-order parameters p_m and q_n do not equal zero. If $m \neq n$ the model will not be able to give instantaneous response to a step in stress or strain.

To emphasize the inherent memory properties of the constitutive model, the relation in Eq. (1) may be written in hereditary form:

$$\sigma(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^t E_{\mathrm{rel}}(t-\tau)\varepsilon(\tau)\,\mathrm{d}\tau\right] \tag{3}$$

where $E_{\mathrm{rel}}(t)$ is the relaxation modulus, i.e., the stress response to unit strain imposed at time $t = 0$. It is assumed that $\varepsilon(t) = 0$ for $t < 0$. The constitutive relation in hereditary form, Eq. (3), is more general than the relation in differential operator form, Eq. (1). To describe the same viscoelastic behavior as Eq. (1) together with the initial condition in Eq. (2) (in the case that $m = n = N$), the relaxation modulus should be chosen as

$$E_{\mathrm{rel}}(t) = E - \sum_{i=1}^{N} \Delta E_i (1 - e^{-t/b_i}) \tag{4}$$

Here ΔE_i and b_i are the relaxation strength and the relaxation time corresponding to the ith dissipative mechanism. The correspondence between Eqs. (1) and (3) can be shown by applying the Laplace transform to the equations (see Ref. 8). Note that Eq. (3) does not need explicit initial conditions because they are automatically accounted for.

The constitutive relation in Eq. (3) can be rewritten into a form that is more convenient in the case of discontinuities in the strain history. The form is obtained by carrying out the differentiation

$$\sigma(t) = E\varepsilon(t) - \sum_{i=1}^{N}\left[\Delta E_i \int_0^t f_i(t-\tau)\varepsilon(t)\,\mathrm{d}\tau\right] \tag{5}$$

where $E = E_{\mathrm{rel}}(0)$ and $f_i(t)$ is the memory kernel corresponding to the ith dissipative mechanism. The memory kernel is related to the relaxation modulus as

$$-\frac{\mathrm{d}}{\mathrm{d}t} E_{\mathrm{rel}}(t) = \sum_{i=1}^{N} \Delta E_i f_i(t) \tag{6}$$

Constitutive equations that can be reduced to the form of Eq. (5) clearly represent relations where the current stress does not solely depend on the current strain but also on the previous strain history.

By using fractional derivative operators in the constitutive equation instead of integer derivative operators as in Eq. (1), the number of parameters required to describe the weak frequency dependence of damping over a broad frequency range can be significantly reduced. The reason for this is that the Fourier transform of an integer-order derivative operator is proportional to frequency raised to the integer order, whereas the Fourier transform of a fractional-order derivative operator is proportional to frequency raised to the fractional order. The simplest uniaxial fractional derivative model uses four parameters, and the constitutive equation is often written as

$$\sigma(t) + b^{\alpha}\mathrm{D}^{\alpha}\sigma(t) = E_{\infty}\varepsilon(t) + E b^{\alpha}\mathrm{D}^{\alpha}\varepsilon(t), \qquad 0 < \alpha \leq 1 \tag{7}$$

(Ref. 2). Here E and E_{∞} can be identified as the instantaneous modulus and the long-time modulus, respectively, whereas D^{α} is a fractional derivative operator of order α and b is a relaxation constant. Bagley and Torvik[9] claimed that Eq. (7) is both thermodynamically and mechanically admissible under stationary harmonic excitation if the parameters are subjected to the following restrictions:

$$E \geq E_{\infty} > 0, \qquad b > 0 \tag{8}$$

A suitable definition of the fractional derivative of a function $x(t)$ is

$$\mathrm{D}^{\alpha}x(t) \equiv \frac{\mathrm{d}^N}{\mathrm{d}t^N}\left[\frac{1}{\Gamma(\rho)}\int_0^t (t-\tau)^{(\rho-1)}x(\tau)\,\mathrm{d}\tau\right]$$
$$0 < \rho = N - \alpha \leq 1 \tag{9}$$

(Ref. 10), where N is an integer. Specializing the definition in Eq. (9) to the interval $\alpha \in (0, 1)$, which is of interest for application to viscoelasticity, yields

$$\mathrm{D}^{\alpha}x(t) \equiv \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^t \frac{x(\tau)}{(t-\tau)^{\alpha}}\,\mathrm{d}\tau\right] \tag{10}$$

In the limit when α tends to one, D^{α} tends to the first derivative. The Laplace transform of the fractional-order derivative is

$$\mathcal{L}[\mathrm{D}^{\alpha}x(t)](s) = s^{\alpha}\mathcal{L}[x(t)](s) - \sum_{k=0}^{n-1} s^k\,\mathrm{D}^{\alpha-1-k}x(0^+) \tag{11}$$

(Ref. 10), where n is an integer such that $n - 1 < \alpha \leq n$. Note the occurrence of initial conditions of fractional order in Eq. (11).

Equation (7) is, with the present definition of fractional differentiation, a differential equation and it needs initial conditions. Enelund and Olsson[4] state that the appropriate condition is

$$\mathrm{D}^{\alpha-1}\sigma(0^+) = E\mathrm{D}^{\alpha-1}\varepsilon(0^+) \tag{12}$$

which may be seen by applying the Laplace transform to Eq. (7). The condition in Eq. (12) is a fractionally integrated form of Hooke's law, i.e., it is a weaker condition than Hooke's law. Note that the initial condition allows for singularities in the strain. However, the work done to deform the material should be bounded for states where the linear viscoelastic model applies, which disqualifies unbounded strains. Having this in mind, the initial condition can be taken as

$$\sigma(0^+) = E\varepsilon(0^+) \tag{13}$$

which implies the condition in Eq. (12).

One way to avoid explicit use of initial conditions is to use a distributional fractional derivative. Gel'fand and Shilov[11] use the following convolution to define the fractional derivative:

$$\bar{\mathrm{D}}^{\alpha}[x(t)] = (x * \Phi_{-\alpha})(t) \tag{14}$$

where

$$\Phi_{-\alpha}(t) = \begin{cases} \dfrac{t^{-1-\alpha}}{\Gamma(-\alpha)} & t > 0 \\ 0 & t < 0 \end{cases} \tag{15}$$

The integral in Eq. (14) is (for $\alpha > 0$) normally divergent and has to be suitably regularized. By replacing α by $-\alpha$ in Eq. (14) the fractional integral of order α of the function x is obtained. The integral (14) is then convergent for sufficiently regular functions x.

The fractional calculus model of viscoelasticity as formulated in Eq. (7) together with the initial condition in Eq. (12) can be

formulated in terms of a convolution integral; cf. Eq. (5). The kernel is then of Mittag–Leffler type (Ref. 3):

$$\sigma(t) = E\varepsilon(t) - \Delta E(f_\alpha * \varepsilon)(t) = E\varepsilon(t) - \Delta E\int_0^t f_\alpha(t-\tau)\varepsilon(\tau)\,\mathrm{d}\tau \tag{16a}$$

with

$$f_\alpha(t) = -\frac{\mathrm{d}E_\alpha[-(t/b)^\alpha]}{\mathrm{d}t} \tag{16b}$$

Here $\Delta E = E - E_\infty$ and E_α is the α-order Mittag–Leffler function, which is defined as

$$E_\alpha(t) = \sum_{k=0}^{\infty}\frac{t^k}{\Gamma(1+\alpha k)} \tag{17}$$

(Ref. 12).

III. Fractional Integral Formulations

As indicated in the preceding section, there is, in general, more than one way to write the constitutive relation of a viscoelastic material. What is of interest here is specifically the relative advantages of the possible formulations of the constitutive relation using fractional calculus. Two different ways of formulating this relation have already been mentioned, namely, in terms of fractional derivative operators [see Eq. (7)] and in terms of a convolution integral [see Eq. (16)]. Obviously, an alternative to using the fractional derivative formulation in Eq. (7) would be to formulate a relation involving fractional integrals. Glöckle and Nonnenmacher[6] have suggested the following law (here rewritten into the notation of the present paper):

$$\sigma(t) - \sigma_0 + (1/b^\alpha)\mathrm{D}^{-\alpha}\sigma(t) = E_\infty(1/b^\beta)\mathrm{D}^{-\beta}\varepsilon(t) + E[\varepsilon(t) - \varepsilon_0] \tag{18}$$

where, in fact, thermodynamical considerations imply $\alpha = \beta$. In Eq. (18), σ_0 and ε_0 are integration constants. Note how Glöckle and Nonnenmacher arrive at Eq. (18). In brief, the authors start from the standard linear solid model and perform (in order) the two operations of integrating and generalizing to noninteger integration, two operations that do not, in general, commute. Another suggestion for a fractional integral model would be to formally invert the fractional derivatives in Eq. (7). This is equivalent to putting the parameters σ_0 and ε_0 to zero in Eq. (18), while imposing the restriction $\alpha = \beta$:

$$\sigma(t) + (1/b^\alpha)\mathrm{D}^{-\alpha}\sigma(t) = E\varepsilon(t) + (1/b^\alpha)E_\infty\mathrm{D}^{-\alpha}\varepsilon(t) \tag{19}$$

Note that applying the α-order derivative to Eq. (19), assuming sufficient regularity of each term, Eq. (7) is obtained, but that the reverse is not possible, in general. (This is why it was called a formal inversion.) Also note that applying the α-order derivative to Eq. (18) with $\alpha = \beta$ does not, in general, yield Eq. (7).

The model in Eq. (18) has been criticized by Friedrich,[7] who argues that it compares unfavorably with another model involving fractional derivatives. From Ref. 7 it may be concluded that, when the restriction $\alpha = \beta$ is imposed on Eq. (18), the model is thermodynamically admissible. The model preferred by Friedrich[7] has one parameter more than the model in Eq. (18) has, when $\alpha = \beta$, and, therefore, gives a better fit to experimental data. There are, however, as will be discussed subsequently, a number of advantages to be gained from using fractional integrals rather than fractional derivatives in the constitutive models.

A. Uniqueness

Consider the three models given by Eqs. (7), (16), and (19). To make a comparison of the three formulations, one can begin by observing that Eq. (7) differs from the other two in the respect that it [like Eq. (1)] requires to be supplemented by initial conditions to represent a unique relation between the stress and the strain history. [Equivalently, one can say that the differential operator in Eq. (7) must be interpreted as the distributional derivative to yield a unique relation between stress and strain.] In this respect it differs from both Eqs. (16) and (19) because these, for a given strain history, yield a unique stress. For Eq. (16) this is obvious. For Eq. (19), considered in a suitable functional framework, the uniqueness follows from Fredholm theory because Eq. (19) can then be interpreted as a Fredholm integral equation of the second kind (with a sufficiently regular kernel).

B. Regularity Requirements

The formulation in Eq. (7) imposes more restrictive conditions on the strain and stress for the occurring integrals to be well defined than do Eqs. (16) and (19). This is essentially the duality between the regularity of the kernel and that of the function with which it is convolved. Because the kernels occurring in Eqs. (16) and (19) are less singular than the kernel occurring in Eq. (7), the local regularity requirements on the strain are less severe in Eqs. (16) and (19) than in Eq. (7). The most direct way of seeing this is by considering the distributional definition of the fractional differentiation and integration operators in Gel'fand and Shilov.[11] The regularization scheme in this reference puts higher requirements on the smoothness properties of the functions on which the fractional differentiation operator acts, than in the case of fractional integration. Note, however, that the mathematical smoothness requirements are in a sense too permissive. Physical considerations are more restrictive than the mathematical in the sense that the strains must all be bounded. Too large strains would, of course, move the problem altogether out of the linear regime.

IV. Structural Equations of Motion

Considering a one-degree-of-freedom system consisting of a discrete mass and a viscoelastic spring governed by a fractional calculus law, there are, corresponding to the various formulations of the constitutive law, a number of different ways of writing the equation of motion. First, the following replacements in Eqs. (7), (16), and (19) are made.

Spring displacement

$$\varepsilon(t) \longrightarrow u(t) \tag{20a}$$

Spring force

$$\sigma(t) \longrightarrow r(t) \tag{20b}$$

Spring constant

$$E \longrightarrow k \tag{20c}$$

Long-time spring constant

$$E_\infty \longrightarrow k_\infty \tag{20d}$$

The equation of motion is, of course, Newton's second law for the one-degree-of-freedom system,

$$M\mathrm{D}^2u(t) = R(t) - r(t) \tag{21}$$

where $R(t)$ is the applied force and M is the mass. In a previous paper,[4] a discussion of certain properties of the fractional derivative model for damping concerned the problems of what initial conditions should apply. When one of the constitutive models in Eqs. (7), (16), and (19) is combined with Newton's second law one of the following equations is obtained:

$$M\mathrm{D}^2u(t) + ku(t) - (k - k_\infty)(f_\alpha * u)(t) = R(t) \qquad t > 0 \tag{22a}$$

$$b^\alpha M\mathrm{D}^\alpha\mathrm{D}^2u(t) + M\mathrm{D}^2u(t) + kb^\alpha\mathrm{D}^\alpha u(t) + k_\infty u(t) = b^\alpha\mathrm{D}^\alpha R(t) + R(t), \quad t > 0 \tag{22b}$$

$$b^\alpha M\mathrm{D}^{2+\alpha}u(t) + M\mathrm{D}^2u(t) + kb^\alpha\mathrm{D}^\alpha u(t) + k_\infty u(t) = b^\alpha\mathrm{D}^\alpha R(t) + R(t), \quad t > 0 \tag{22c}$$

$$M\mathrm{D}^2u(t) + (M/b^\alpha)\mathrm{D}^{-\alpha}\mathrm{D}^2u(t) + ku(t) + (k_\infty/b^\alpha)\mathrm{D}^{-\alpha}u(t) = R(t) + (1/b^\alpha)\mathrm{D}^{-\alpha}R(t), \qquad t > 0 \tag{22d}$$

The first of these equations makes use of the memory kernel corresponding to the fractional derivative model. Here the constitutive law used is on the Boltzmann form as in Eq. (16). The second and third equations have been derived by applying fractional derivatives to Newton's second law [Eq. (21)] to eliminate the spring force from this equation by means of the constitutive law in the Eq. (7) form. The difference between Eqs. (22b) and (22c) is that in Eq. (22c) the initial displacement $u(0^+)$ and the initial velocity $\mathrm{D}u(0^+)$ are taken to be zero and the composition rule for fractional derivatives is used (see Ref. 10). The fourth equation has been derived by means of Eq. (19), the integral form of the constitutive law.

A. Initial Conditions

Taking the Laplace transform of each of the equations of motion indicates that the sets of initial conditions appropriate for Eqs. (22a–22c) are

$$[u(0^+),\ \mathrm{D}u(0^+)] \tag{23a}$$

$$[\mathrm{D}^{\alpha-1}u(0^+),\ u(0^+),\ \mathrm{D}u(0^+),\ \mathrm{D}^{\alpha-1}\mathrm{D}^2u(0^+)] \tag{23b}$$

$$[\mathrm{D}^{\alpha-1}u(0^+),\ u(0^+),\ \mathrm{D}^{\alpha}u(0^+),\ \mathrm{D}u(0^+),\ \mathrm{D}^{\alpha+1}u(0^+)] \tag{23c}$$

respectively. Physical intuition seems to favor the set in Eq. (23a) and, thus, the formulation (22a). For a fuller discussion, see Refs. 4 and 13. In the latter reference it is stated that, in mechanical applications, "irritation arose concerning the physical interpretation, justification and verification" of the fractional initial conditions.

However, when turning to consider Eq. (22d), which superficially is so much like Eq. (22b), it is seen that the set of initial conditions required is the same as for Eq. (22a), i.e., the set in Eq. (23a). By the simple remedy of using fractional integrals rather than fractional derivatives, the required initial conditions are reduced to physical ones. (The initial conditions are physical both in the sense that they are sufficient to determine the subsequent behavior of the system and that they are straightforward to measure.)

B. Numerical Properties

The difference between Eqs. (22b) and (22c) is that in Eq. (22c) the initial displacement $u(0^+)$ and the initial velocity $\mathrm{D}u(0^+)$ are taken to be zero. If the displacement u and its derivatives are zero for negative times, $t \leq 0$, Eq. (22c) may be solved in the Fourier domain. The Fourier transform $\mathcal{F}[\mathrm{D}^{\alpha}u(t)](\omega)$ of the derivative $\mathrm{D}^{\alpha}u(t)$ may then be written

$$\mathcal{F}[\mathrm{D}^{\alpha}u(t)](\omega) = (i\omega)^{\alpha}\mathcal{F}[u(t)](\omega) \tag{24}$$

Here $(i\omega)^{\alpha}$ is the principal root and the Fourier transform is defined as

$$\mathcal{F}[u(t)](\omega) = \int_{-\infty}^{\infty} u(t)e^{-i\omega t}\,\mathrm{d}t \tag{25}$$

After taking the Fourier transform of Eq. (22c), the Fourier transform of the displacement may be solved. The displacement can be transformed back into the time domain numerically. Note that the restrictions on the initial conditions also give restrictions on the loads that are applied to the system.

Another way of solving Eq. (22c), if the initial displacement and all of its initial derivatives are zero, is to use a modal synthesis method as described in Ref. 14. The order α of the derivatives is then supposed to be (or to be approximated as) a rational number, $\alpha = m/n$. By introducing a state-space vector containing the displacement and its derivatives of order $1/n,\ 2/n,\ \ldots,\ (2n+m-1)/n$, the differential equation (22c) of order $2+\alpha$ is reduced to $2n+m$ differential equations of order $1/n$, which are more easily solved.

Consider now Eq. (22a). In Ref. 4, an efficient numerical scheme for solving this equation, subject to initial conditions according to Eq. (23a), was given. One major advantage of that scheme, as compared to an approach where the convolution must be evaluated from a computation of the memory kernel, is the complete avoidance of any computation of f_{α}. The scheme takes advantage of the following property of the convolution term in Eq. (22a):

$$\mathrm{D}^{\alpha}(f_{\alpha} * u)(t) = -(1/b^{\alpha})(f_{\alpha} * u)(t) + (1/b^{\alpha})u(t) \tag{26}$$

This time evolution property, combined with a truncated version of Grünwald's definition of the fractional derivative together with the backward Euler rule and Newmark's method for integration of the dynamic response, yields an efficient scheme for solving Eq. (22a) under conditions in Eq. (23a), as described in Ref. 4.

V. Algorithm

When discretizing Eq. (22d), the problematic terms are the ones containing $\mathrm{D}^{-\alpha}\mathrm{D}^2u$ and $\mathrm{D}^{-\alpha}u$. The time evolution of these terms can be expressed by applying a suitable truncation of Grünwald's definition of fractional differintegration, i.e., fractional differentiation or fractional integration, as

$${}^{n+1}(\mathrm{D}^{-\alpha}u) = (\Delta t)^{\alpha}\left[{}^{n+1}u + \sum_{j=1}^{n} B_j(-\alpha)^{n+1-j}u\right] \tag{27a}$$

(Ref. 10) with

$$B_j(\alpha) = \frac{\Gamma(j-\alpha)}{\Gamma(-\alpha)\Gamma(j+1)} \tag{27b}$$

$${}^{n+1}(\mathrm{D}^{-\alpha}\mathrm{D}^2u) = (\Delta t)^{\alpha}\left[{}^{n+1}(\mathrm{D}^2u) + \sum_{j=1}^{n} B_j(-\alpha)^{n+1-j}(\mathrm{D}^2u)\right] \tag{27c}$$

It is then assumed that the spacing in time is uniform, i.e., ${}^{n}u(t) = u(n\Delta t)$. The calculation of $B_j(\alpha)$ is simplified by the recursion formula

$$\frac{\Gamma(j-\alpha)}{\Gamma(j+1)} = \frac{(j-1-\alpha)}{j}\,\frac{\Gamma(j-1-\alpha)}{\Gamma(j)} \tag{28}$$

Note that it is necessary to compute and store all values ${}^{m}(\mathrm{D}^{-\alpha}u)$ and ${}^{m}(\mathrm{D}^{-\alpha}\mathrm{D}^2u)$ for $m = 0, 1, 2, \ldots, n$. Unless $\mathrm{D}^{-\alpha}R(t)$ can be computed analytically, it can be discretized from Grünwald's definition as

$${}^{n+1}(\mathrm{D}^{-\alpha}R) = (\Delta t)^{\alpha}\left[{}^{n+1}R + \sum_{j=1}^{n} B_j(-\alpha)^{n+1-j}R\right] \tag{29}$$

Using a nearly explicit Newmark's method for time integration, together with initial conditions on ${}^{0}u$ and ${}^{0}(\mathrm{D}u)$, the displacement solution at time ${}^{n+1}t = (n+1)\Delta t$ is obtained as

$${}^{n+1}u = {}^{n}u + \Delta t\,{}^{n}(\mathrm{D}u) + (\Delta t^2/2)\,{}^{n}(\mathrm{D}^2u) \tag{30a}$$

and the velocity is obtained from

$${}^{n+1}(\mathrm{D}u) = {}^{n}(\mathrm{D}u) + (\Delta t/2)[{}^{n+1}(\mathrm{D}^2u) + {}^{n}(\mathrm{D}^2u)] \tag{30b}$$

The updated acceleration ${}^{n+1}(\mathrm{D}^2u)$ is obtained from a discretized version of Eq. (22d), together with the expression for ${}^{n+1}(\mathrm{D}^{-\alpha}\mathrm{D}^2u)$ in Eq. (27c), as

$$\begin{aligned}{}^{n+1}(\mathrm{D}^2u) = \frac{1}{1+(\Delta t/b)^{\alpha}}M^{-1}\Bigg[&{}^{n+1}R + \frac{1}{b^{\alpha}}{}^{n+1}(\mathrm{D}^{-\alpha}R)\\ &- M\left(\frac{\Delta t}{b}\right)^{\alpha}\sum_{j=1}^{n} B_j(-\alpha)^{n+1-j}(\mathrm{D}^2u)\\ &- k\,{}^{n+1}u - \frac{k}{b^{\alpha}}{}^{n+1}(\mathrm{D}^{-\alpha}u)\Bigg]\end{aligned} \tag{31}$$

where ${}^{n+1}(\mathrm{D}^{-\alpha}u)$ is obtained from Eq. (27a). The initial acceleration ${}^{0}(\mathrm{D}^2u)$ is computed from

$${}^{0}(\mathrm{D}^2u) = {}^{0}R/M \tag{32}$$

because the spring force is initially equal to zero.

The nearly explicit Newmark's method is known to be conditionally stable. In the purely elastic or undamped case, the critical time step is given by

$$\Delta t_{\text{crit}} = 2/\omega_{\max} \tag{33}$$

where $\omega_{\max}$ is the highest elastic or undamped eigenfrequency. This time step is believed to be conservative for a viscoelastic structure because the structure weakens with increasing time, which results in decreasing $\omega_{\max}$. For the present one-degree-of-freedom system, the critical time step is taken to be $\Delta t_{\text{crit}} = 2/\sqrt{(k/M)}$. Note that k is the instantaneous or unrelaxed stiffness.

A suitable scheme for calculating the unknowns is as follows. The initial acceleration ${}^{0}(\mathrm{D}^2 u)$ is obtained from Eq. (32). The updated displacement ${}^{n+1}u$ is directly obtained from Eq. (30a). Insertion of ${}^{n+1}u$ into Eq. (27a) gives ${}^{n+1}(\mathrm{D}^{-\alpha}u)$, and the updated acceleration is obtained from Eq. (31). Using this, the updated velocity ${}^{n+1}(\mathrm{D}u)$ is obtained from Eq. (30b), and a new time step can be taken.

VI. Numerical Examples

Some numerical examples are presented to show the ability and the purpose of the present fractional integral formulation and the accompanying algorithm for time integration of transient responses.

Consider the case where $\alpha = \frac{1}{2}$ and an impulsive loading, i.e., $R(t) = F_0\delta(t)$. As far as Eq. (22d) goes, this is equivalent to specifying zero load but a nonvanishing initial velocity and zero displacement, i.e.,

$$\begin{aligned} R(t) &= 0, \qquad t > 0 \\ u(0^+) &= 0 \\ \mathrm{D}u(0^+) &= v_0 = F_0/M \end{aligned} \tag{34}$$

In this case, the first few terms of a series expansion, valid for small times t of the displacement solution, obtained in the Laplace domain, are

$$u(t) = v_0 t - \frac{v_0 k t^3}{6M} + \frac{16 v_0 (k - k_\infty) t^{\frac{7}{2}}}{105 M b^{\frac{1}{2}} \sqrt{\pi}} + \mathcal{O}(t^4) \tag{35}$$

(Refs. 4 and 5). Figure 1 shows the numerically obtained displacement solution and the time series expansion of the displacement solution in the case of $\alpha = \frac{1}{2}$ and unit impulsive loading as described in Eq. (34). The agreement between the numerical solution and the time series expansion is good for small times, indicating that the algorithm is accurate. For larger times, the solutions deviate because the time series expansion is no longer valid. Figure 2 shows numerical displacement solutions in the case of a unit impulsive loading as described in Eq. (34). The influence of different values of the fractional derivative exponent α ($\alpha = \frac{1}{3}, \frac{1}{2}$, and $\frac{2}{3}$) is displayed. Figure 3 shows numerical displacement solutions in the case of a unit step load applied at $t = 0$ and homogeneous initial conditions [i.e., $u(0^+) = \mathrm{D}u(0^+) = 0$]. Also here, the influence of different values of the fractional derivative exponent α ($\alpha = \frac{1}{3}, \frac{1}{2}$, and $\frac{2}{3}$) is displayed.

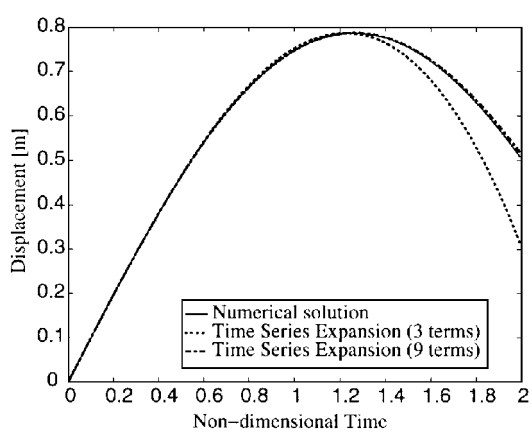

Fig. 1 Displacement vs nondimensional time t/b for fractionally damped one-degree-of-freedom system subjected to a unit impulse load at time $t = 0$; time series expansions are valid for small times.

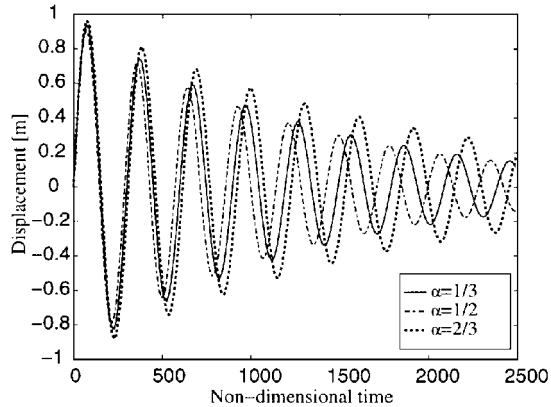

Fig. 2 Displacement vs nondimensional time for a fractionally damped one-degree-of-freedom system subjected to a unit impulse load at time $t = 0$.

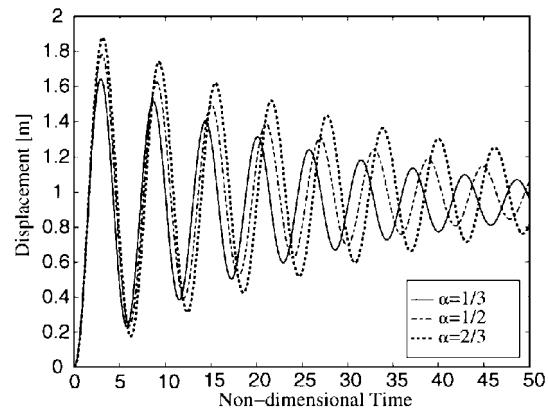

Fig. 3 Displacement vs nondimensional time for a fractionally damped one-degree-of-freedom system subjected to a unit step load at time $t = 0$.

Consider now the case of an eigenfunctional load and homogenous initial conditions, i.e.,

$$R(t) = C_0 f_\alpha(t), \qquad u(0^+) = 0, \qquad \mathrm{D}u(0^+) = 0 \tag{36}$$

Here f_α is the same as in Eq. (16b) and C_0 is the amplitude with dimension of force times time (Newton-seconds). The right-hand sides of Eqs. (22b) and (22c) vanish for this load, and homogenous initial conditions on all of the quantities in Eqs. (23b) and (23c) would yield the trivial solution. However, this is not the case with Eqs. (22a) and (22d). They both predict nontrivial displacements for this loading case. The right-hand side of Eq. (22d) can be expressed as

$$\begin{aligned} R(t) + (1/b^\alpha)\mathrm{D}^{-\alpha}R(t) &= C_0(1/b^\alpha)\mathrm{D}^{-\alpha}\delta(t) \\ &= C_0(1/b^\alpha)[t^{\alpha-1}/\Gamma(\alpha)] \end{aligned} \tag{37}$$

For the eigenfunctional load, the first few terms in a series expansion of the displacement solution, valid for small times, obtained in the Laplace domain, in the case of $\alpha = \frac{1}{2}$ are

$$u(t) = \frac{4C_0 t^{\frac{3}{2}}}{3b^{\frac{1}{2}} M \sqrt{\pi}} - \frac{C_0 t^2}{2bM} + \frac{8C_0 t^{\frac{5}{2}}}{15 b^{\frac{3}{2}} M \sqrt{\pi}} + \mathcal{O}(t^3) \tag{38}$$

Figure 4 shows the numerically obtained displacement solution and the time series expansion of the displacement solution in the case of $\alpha = \frac{1}{2}$ and unit eigenfunctional loading as described in Eq. (36). The agreement between the numerical solution and the time series expansion is good for small times, where the time series expansion is valid.

Special consideration must be taken when computing the initial acceleration from Eq. (32), because the eigenfunctional load is unbounded at $t = 0$. The singular part of the load is treated as an

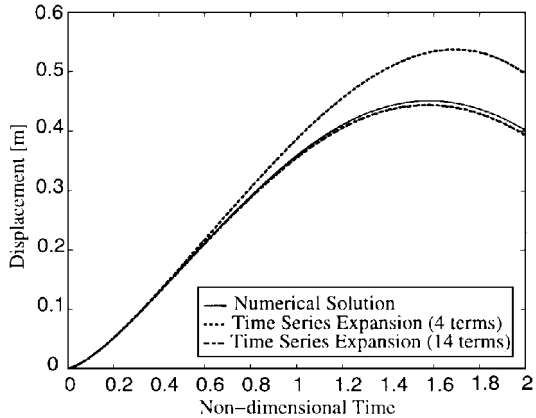

Fig. 4 Displacement vs nondimensional time t/b for fractionally damped one-degree-of-freedom system subjected to a unit eigenfunctional load at time $t = 0$; time series expansions are valid for small times.

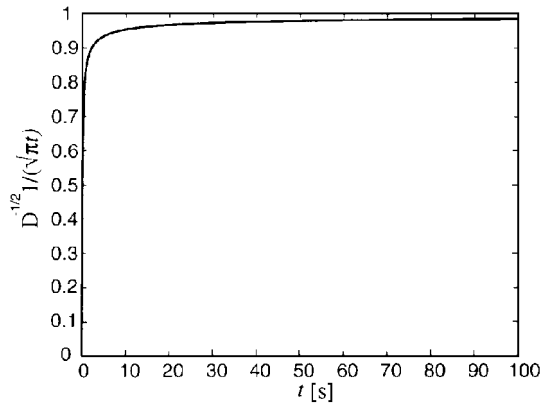

Fig. 5 Numerical half-order integral of function $1/\sqrt{(\pi t)}$ calculated by the Grünwald algorithm utilizing a maximum of 1000 data points; exact value of the integral is 1 for all t.

impulse, and the value of the applied load at time $t = 0$ is regularized so that the impulse will be correct. To obtain a finite value of $R(t)$ at time zero, ${}^0R$ is determined as

$$ {}^0R = \frac{2}{\Delta t}\int_0^{\Delta t} R(t)\,\mathrm{d}t - {}^1R \tag{39} $$

where ${}^1R = R(\Delta t)$, which is finite and known beforehand.

Numerical values used in the preceding examples are $M = 1$ kg, $k = 2$ N/m, and $k_\infty = 1$ N/m, whereas $b = 1$ s in Figs. 1 and 4 and $b = 0.02$ s in Figs. 2 and 3. The time steps chosen are sufficiently smaller than the critical time step in the elastic case. In Figs. 2 and 3, $\Delta t = 0.1\Delta t_{\text{crit}}$ is used based on accuracy considerations.

Similar examples have been studied in Refs. 4 and 5 when using a convolution integral form of the fractional calculus model of viscoelasticity [cf. Eq. (22a)]. The calculated responses in the present study are in close agreement with the ones reported in Refs. 4 and 5. However, when using the present formulation, the time histories of both the displacement and the acceleration have to be stored and included in each time step. Unless the fractional integral of the applied load function can be computed analytically it has to be stored as well. In the case of the convolution formulation, it is sufficient to store the time history of the convolution term and include it in each time step. This makes the convolution form much more computationally efficient.

The numerical differintegration of unbounded functions seems to be troublesome. It is stated in Ref. 10 that the Grünwald algorithm [Eq. (27a)] is unique in not utilizing the value of the function in the singular point (here $t = 0$) when obtaining the numerical differintegration. Therefore, it should be suitable for computing fractional-order derivatives and integrals of unbounded functions. However, it seems that the Grünwald algorithm converges very slowly in the case of fractional differintegration of unbounded functions, although it is very suitable for differintegration of bounded functions. Consider, e.g., the half-order integral of $1/\sqrt{(\pi t)}$. The exact value of this half-order integral is

$$ \mathrm{D}^{-\frac{1}{2}}(1/\sqrt{\pi t}) = 1 \tag{40} $$

Figure 5 shows the approximation of the half-order integral of $1/\sqrt{(\pi t)}$ by the Grünwald algorithm utilizing a maximum of 1000 data points in the summation [cf. Eq. (27a)]. The numerically obtained integral is less than the exact value, which is equal to 1 for all $t > 0$. This example illustrates that there are difficulties with the numerical calculations of fractional differintegration of unbounded functions. If the load contains unbounded functions, fractional-order integrals or derivatives of unbounded functions have to be computed for the present integral formulation in Eq. (22d) and for the formulations in Eqs. (22b) and (22c). However, if using the convolution formulation, as in Eq. (22a), fractional differintegration of unbounded functions is avoided.

VII. Concluding Remarks

A fractional integral constitutive equation of viscoelasticity is presented. This equation represents a unique relation between stress and strain, i.e., no initial conditions need to be imposed. This fractional integral model describes the same material behavior as the fractional derivative model of viscoelasticity together with suitable initial conditions. The fractional integral model compares favorably to the fractional derivative model when incorporated directly into structural equations of motion, in that it does not raise the order of the equations. This means that as far as the resulting equations of motion are concerned, only physical initial conditions are required.

A numerical scheme for time integration of responses for viscoelastic one-degree-of-freedom systems governed by the fractional integral constitutive equation is presented. The implementation reveals that the fractional integral model is in many respect similar to the Boltzmann or convolution integral model [cf. Eqs. (16) and (22a)]. A drawback of the fractional integral model is that it requires that the time histories of the acceleration and the displacement are stored and included in each time increment. Further, if the fractional integral of the applied load function cannot be obtained analytically, the load history has to be stored as well. If the convolution integral form is used, only the time history of the convolution term needs to be stored. It may also be noted that the convolution integral model handles singular loads better than does the fractional integral model, because there is no fractional integral operator applied to the load.

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